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Renormalization group flow of reduced string actions

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Abstract: It has been argued that certain reduced actions play a role in AdS/CFT when comparing fast moving strings to long single trace operators in gauge theories. Such actions arise in two ways: as a limit of the string action and as a description of long single trace field theory operators. They are non-relativistic sigma models with the target space usually being a Kähler manifold. They are non-renormalizable and need a cut-off in the wave-length. If the total spin (or charge) contained in a minimal wavelength is large compared to one, the system behaves semiclassically and an expansion in loops is meaningful.
In this paper we apply the renormalization group procedure to such actions and find, at one-loop, that the Kähler potential flows in the infrared to a Kähler-Einstein one. Therefore, in this context, the anomalous dimensions of long operators are determined by a fixed point. This suggests that certain features of the large N-limit might be independent of the detailed properties of a gauge theory.

Keywords: spin chains, string theory, renormalization group.

Contents

1. Introduction

It has since long been suspected that four dimensional confining theories have a dual string description in the large-N limit [1]. Relatively recently a precise example of a duality between a gauge theory and a string theory was established through the AdS/CFT correspondence [2]. Following that, Berenstein, Maldacena and Nastase [3] show a matching between certain operators in the boundary theory and excited strings in the bulk. Such relation turned out to be part of a more general relation between semi-classical strings in the bulk [4] and certain operators in the boundary [1]. In another development, Minahan and Zarembo [8] observed that the one-loop anomalous dimension of operators composed of scalars in $\mathcal{N} = 4$ SYM theory follows from solving and integrable spin chain[2]. This allowed the authors of [10, 11] to make a much more detailed comparison between particular string solutions and operators in the gauge theory.

It was later suggested [12] that a classical sigma model action follows from considering the low energy excitations of the spin chain. It turned out that such action agrees

[1]See the recent reviews [5, 6, 7] for a summary with a complete set of references.

[2]In QCD the relation between spin chains and anomalous dimensions had already been noted in [9].

with a particular limit of the sigma model action that describes the propagation of the strings in the bulk. This idea was extended to other sectors including fermions and open strings in [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] and to two loops in [25]. In [26, 27] the relation to the Bethe ansatz approach was clarified. Further, similar possibilities were found in certain subsectors of QCD [28]. A different but related approach to the relation between the string and Yang-Mills operators has been pursued in [29, 30, 31] where a geometric description in terms of light like world-sheets was found. In that respect see also [32]. A similar picture seems to appear also in other sector [33].

Recently a large class of new examples of the AdS/CFT correspondence was found. They are of the form $AdS_5 \times X_5$ where X_5 is a Sasaki-Einstein manifold. In the new examples, X_5 is one of the so called $Y^{p,q}$ and $L^{p,q|r}$ manifolds whose metrics where found in [34, 35] and [36], and the dual gauge theories in [37] and [38].

It was natural to see if the methods of the reduced action could be applied to such case. This was considered in [39] for the $Y^{p,q}$ case. The field theories are strongly coupled and therefore no field theory calculations were possible. However it was suspected that the string action only captures generic properties of the operators. That was partially confirmed by using a simplified model for the operators which, in the classical limit, was described by an action similar (but not equal) to the action derived from the string side. It was further argued that in the infrared, the action of the simplified model should flow to the one obtained from the string side. This was based on the fact that the metric in the string side satisfied the Einstein equations (with cosmological constant) and the idea that the Einstein equation was a natural candidate for an equation determining an infrared fixed point of the action.

In this paper we analyze this problem for a class of actions that depend on a Kähler potential in an n complex dimensional manifold. They include most of the known examples of reduced actions.

By doing a one-loop computation we confirm that the reduced action obtained from the string side is at a fixed point. However for an action to flow towards that fixed point, some conditions have to be met which do not seem to be satisfied by the simplified model of [39]. This means that, although the basic idea is correct, in the sense that a large class of models flow to the one of interest, the particular one considered in [39] does not seem to be in that class. It would be interesting to see if there is a simple model that actually have the correct properties.

Before proceeding we should note that, in the $SU(2)$ case, loop corrections were studied in several papers [40] (see also [25]). The work here could be useful in finding an expression for the one-loop corrections independent of the particular classical solution considered. This calculation of higher order terms in the effective action is interesting but will not be attempted here.

Another comment is about the cut-off. In the spin chain side there is a natural cut-off given by the lattice spacing. In the string side although there is no cut-off in the original action, the reduced action also has a cut-off that determines its regime of validity. This follows already from the BMN analysis which considers an ultrarelativistic string. If the momentum J of the string is large and fixed, there is a maximum possible wave number for the string excitations, otherwise the mass of the string will be large and the ultrarelativistic approximation no longer valid.

The organization of this paper is as follows: in section §2 we review some properties of the reduced action and give the motivation for the calculation. In section §3 we study the action and analyze its scaling properties. In section §4 we use Wilson's renormalization group approach at one loop to compute the flow of the action to the infrared. We end in section §5 by giving an example. Finally we give our conclusions in section §6.

2. Reduced actions

In [12] (see also [25]), it was argued that the action (which we write here in Euclidean space):

$$S = -i\mu \int \cos\theta\dot{\phi} + \frac{\lambda}{2}\int \left[(\partial_\sigma\theta)^2 + \sin^2\theta(\partial_\sigma\phi)^2\right] \tag{2.1}$$

could be used to compute the anomalous dimensions of long operators in the $SU(2)$ sector of $\mathcal{N} = 4$ SYM. The same action appears as a limit of the string action moving in the bulk. This action describes the motion of spins pointing parallel to the direction $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ subjected to a ferromagnetic interaction that tends to put them parallel to each other. Furthermore, μ is the spin of each site, namely it can be considered as a spin density, and λ is a coupling constant that determines the strength of the interaction. The coordinate σ is integrated from 0 to the length of the chain L. This means that μL is the maximum total spin, namely the spin of the ferromagnetic ground state.

This action can be put in the generic form (generalized to d spatial dimensions):

$$S = \mu\int d^dx\,dt\,\left(\dot{z}^\mu\partial_\mu K - \dot{\bar{z}}^{\bar{\mu}}\partial_{\bar{\mu}}K\right) + \lambda\int d^dx\,dt\,\partial_{\mu\bar{\nu}}K\,\partial_j z^\mu\partial_j\bar{z}^{\bar{\nu}} \tag{2.2}$$

where K is a Kähler potential in an n-complex dimensional Kähler manifold. Since in the rest of the paper we study this action in detail it is appropriate to summarize here our notation. The sigma model is a field theory in d spatial dimensions denoted as $i, j, \ldots = 1\ldots d$ and one Euclidean time denoted as t. Time derivatives are denoted with a dot. The target manifold has complex dimension n and holomorphic coordinates

z^μ with $\mu, \nu \ldots = 1 \ldots n$ and anti holomorphic $z^{\bar\mu}$ with $\bar\mu, \bar\nu, \ldots = 1 \ldots n$. We also find convenient at times to consider the manifold as a $2n$ dimensional manifold with coordinates x^α with $\alpha, \beta, \ldots = 1 \ldots 2n$. We also introduce a local frame (vielbein) with n holomorphic one-forms e^a_μ labeled by $a, b, \ldots = 1 \ldots n$ and n anti-holomorphic $e^{\bar a}_{\bar\mu}$ labeled by $\bar a, \bar b \ldots = 1 \ldots n$. We also consider a real frame e^a_α where now $a, b \ldots = 1 \ldots 2n$.

As an example, (2.1) corresponds to $d = 1$, $n = 1$, $K = \ln(1 + z\bar z)$, since a two sphere is a one dimensional complex manifold. Another example appeared in [39] where the same type of action was found for a string moving fast in $AdS_5 \times Y^{p,q}$ in such a way that each piece of the string moves approximately along a BPS geodesic. In the string side, the appearance of a Kähler metric is related to supersymmetry but in the field theory side it seems a feature emerging from the coherent state method of obtaining the classical action. In that case the complex manifold is a submanifold of $CP(n)$ which is the space of states of a quantum system with n discrete states. These states are the ones that can appear in a site of the spin chain. For example in the $SU(2)$ case there are two states and we get $CP(2) = S^2$. In general however the states can be written in terms of a smaller set of parameters which parameterize the vacuum states.

In the case of $Y^{p,q}$, the target manifold of the sigma model has two complex dimensions and $SU(2)$ isometry. We refer the reader to [39] for the particular form of the Kähler potential since that is not essential for what we do in the present paper. To clarify a bit however we just mention that the four dimensional manifold in question is the base of the $Y^{p,q}$ manifold. The manifold $Y^{p,q}$ is five dimensional and can be written as a $U(1)$ fibration over the four dimensional Kähler base. The base has orbifold singularities although the 5-d manifold is regular.

The properties of the action (2.2), for generic K, is the focus of the rest of this paper. We should note that the actual action has an infinite number of terms from which we are considering only the lowest order ones in an expansion in derivatives. This is allowed as long as we are concerned with the lowest energy states. From that point of view we study the flow under renormalization of these terms and ignore the others.

3. Perturbative expansion

In this section we analyze the action (2.2) to understand the generic properties of its perturbative expansion. For this purpose it is convenient to consider the target manifold a $2n$ real manifold (see below eqn.(2.2) for notation) and write the action as

$$S = -i\mu \int d^dx\, dt\, A_\alpha \dot x^\alpha + \frac{\lambda}{2} \int d^dx\, dt\, g_{\alpha\beta} \partial_j x^\alpha \partial_j x^\beta \tag{3.1}$$

where $\alpha = \mu, \bar{\mu}$ and $A_\mu = i\partial_\mu K$, $A_{\bar{\mu}} = -i\partial_{\bar{\mu}} K$, $g_{\mu\bar{\nu}} = \partial_{\mu\bar{\nu}} K$. Since the action is adimensional we obtain the following units

$$[\mu] = \frac{1}{L^d}, \qquad [\lambda] = \frac{1}{TL^{d-2}} \tag{3.2}$$

where L denotes units of length and T of time. We see that μ is a density which in the $SU(2)$ case is the spin density and generically we can call it a charge density. As we see below the theory is not renormalizable so we need to introduce a UV cutoff Λ that we take to have units of momentum ($[\Lambda] = 1/L$). The only adimensional quantity we can construct is $\mu\Lambda^{-d}$. Since Λ^{-1} is the minimal wavelength, $\mu\Lambda^{-d}$ is the total spin or charge contained in the minimal volume we consider. For example, in the $SU(2)$ case, we can think that such elementary volume can be replaced by a single spin of value $S = \mu\Lambda^{-d}$. If $S \gg 1$ each spin (and therefore the whole system) behaves classically which leads us to expect that the loop counting parameter $\hbar$ is given by $\hbar = \frac{1}{S} = \frac{1}{\mu\Lambda^{-d}}$. This means that, as long as the waves we consider only move the spins in big groups, the classical approximation is valid.

Lets us make this more precise. Expanding x^α around a constant background $x^\alpha = x_0^\alpha + \delta x^\alpha$ it is easy to see that we can write a perturbative expansion with a propagator

$$\langle \delta x(k,\omega) \delta x(k,\omega) \rangle \sim \frac{1}{\lambda k^2 + 2i\mu\omega} \tag{3.3}$$

and vertices of two types. One type of vertex is proportional to μ and contains one time derivative and the other is proportional to λ and contains two spatial derivatives. Both can contain arbitrary number of fields. A generic n_l-loop amplitude with n_E external legs, coming from a Feynman diagram with n_V^t vertices of the first type and n_V^σ vertices of the second type is given schematically by:

$$A \sim \lambda^{n_V^\sigma} \mu^{n_V^t} \left[\int d^d k \, d\omega \right]^{n_l} \left[\frac{1}{\lambda k^2 + 2i\mu\omega} \right]^{n_L} \left(k^2\right)^{n_V^\sigma} \omega^{n_V^t} \tag{3.4}$$

where n_L is the number of internal propagators and we suppressed the labels of the momenta. There are n_l independent loop momenta which are integrated. Also the momenta of the propagators are linear combinations of the loop and external momenta. We can now rescale all k and ω as $k \to k/\sqrt{\lambda}$, $\omega \to \omega/\mu$ to get

$$A \sim \frac{1}{\left(\mu\lambda^{\frac{d}{2}}\right)^{n_l}} \left[\int d^d k \, d\omega \right]^{n_l} \left[\frac{1}{k^2 + 2i\omega} \right]^{n_L} \left(k^2\right)^{n_V^\sigma} \omega^{n_V^t} \tag{3.5}$$

To do the integrals we put a cut-off such that $k^4 + \omega^2 \leq \lambda^2\Lambda^4$ (or $\lambda^2 k^4 + \mu^2\omega^2 \leq \lambda^2\Lambda^4$ before the rescaling). If we rescale the external momenta with the cut-off, then, since

$\sqrt{\lambda}\Lambda$ is the only remaining scale in the integral, we can use dimensional analysis to find that

$$A = \frac{1}{\left(\mu\lambda^{\frac{d}{2}}\right)^{n_l}} \left(\sqrt{\lambda}\Lambda\right)^{(d+2)n_l - 2(n_L - n_V)} \Phi\left(\frac{p}{\Lambda}, \frac{\mu}{\lambda}\frac{w}{\Lambda^2}\right) = \frac{\lambda\Lambda^2}{(\mu\Lambda^{-d})^{n_l}} \Phi\left(\frac{p}{\Lambda}, \frac{\mu}{\lambda}\frac{w}{\Lambda^2}\right) \tag{3.6}$$

where we used that the number of propagators n_L is related to the number of vertices and loops by

$$n_l = n_L - n_V^t - n_V^\sigma + 1 \tag{3.7}$$

by simple counting. We also wrote explicitly the dependence on the external momenta and energies (p, w) through a function Φ that should be determined from the actual calculation. We assume here that the external momenta provide an infrared cut-off to possible IR divergences. In the next section we use an alternative Wilsonian approach where one only integrates a thin momentum shell and therefore no IR divergences can appear. So, as expected, we obtain that the loop counting parameter is $\frac{1}{\mu\Lambda^{-d}}$ which is the inverse of the total charge or spin contained in the minimal volume we consider. When that is large, loop corrections are suppressed and the system behaves classically. In the effective action this amplitude contributes terms of the form

$$S_{\text{eff}} \sim \frac{\lambda}{(\mu\Lambda^{-d})^{n_l}} \left(\frac{\mu}{\lambda}\right)^{n_t} \Lambda^{2-2n_t-n_\sigma} \int d^dx\, dt\, h^{j_1\ldots j_{n_\sigma}}_{\alpha_1\ldots\alpha_{n_t},\beta_1\ldots\beta_{n_\sigma}} \dot{x}^{\alpha_1}\ldots\dot{x}^{\alpha_{n_t}} \partial_{j_1}x^{\beta_1}\ldots\partial_{j_{n_\sigma}}x^{\beta_{n_\sigma}} \tag{3.8}$$

where the coefficients $h^{j_1\ldots j_{n_\sigma}}_{\alpha_1\ldots\alpha_{n_t},\beta_1\ldots\beta_{n_\sigma}}$ can be determined by expanding Φ. If the momenta are of order $k \sim \tilde{\Lambda}$ and the energies of order $\omega \sim \frac{\lambda}{\mu}\tilde{\Lambda}^2$ such term is of order

$$S_{\text{eff}} \sim \frac{\lambda\Lambda^2}{(\mu\Lambda^{-d})^{n_l}} \left(\frac{\tilde{\Lambda}}{\Lambda}\right)^{2n_t+n_\sigma} \tag{3.9}$$

which expresses both, the fact that higher loop corrections are suppressed when $(\mu\Lambda^{-d}) \gg 1$ and that higher derivative terms are suppressed at low energy ($\tilde{\Lambda} \ll \Lambda$). The first fact justifies the loop (or semi-classical) expansion and the second the restriction to lowest order terms.

4. One-loop renormalization group

In this section we study the one-loop renormalization group of the action. The renormalization of sigma models has been studied long ago by Honerkamp and collaborators [41]. The idea that the metric is modified by the loop corrections is well known from the work of Friedan [42] and is a cornerstone of string theory (see *e.g.* [43, 44]).

At the one-loop order it is convenient to use Wilson's procedure of integrating a thin momentum shell which was first applied to sigma models by Polyakov [45].

As in the previous section, we write the action (2.2) in Euclidean space as

$$S = -i\mu \int d^d x \, dt \, A_\alpha \dot{x}^\alpha + \frac{\lambda}{2} \int d^d x \, dt \, g_{\alpha\beta} \partial_j x^\alpha \partial_j x^\beta \tag{4.1}$$

where $\alpha = \mu, \bar{\mu}$ runs over holomorphic and antiholomorphic indices and:

$$A_\mu = i\partial_\mu K \tag{4.2}$$

$$A_{\bar{\mu}} = -i\partial_{\bar{\mu}} K \tag{4.3}$$

$$g_{\mu\bar{\nu}} = \partial_{\mu\bar{\nu}} K \tag{4.4}$$

The action has a cut-off Λ determining the maximum spatial momentum k of the fields. To study the renormalization group we do three steps:

- First we divide the fields into slow ($k < \Lambda e^{-b}$) and fast ($\Lambda e^{-b} < k < \Lambda$) and integrate out the fast modes getting a theory with cut-off Λe^{-b}.
- Then we rescale the coordinates so that the cut-off goes back to being Λ.
- Finally we rescale K to normalize the target metric.

For the first step we should do the separation in fast and slow modes in a generally covariant way. This is not only to get more elegant expressions but also because, in general, to parameterize the target manifold, we need to introduce different coordinate patches. Coordinates in different patches are related by a coordinate transformation and therefore, if our procedure is not invariant under those, the resulting action will not be well defined. In the case of a complex manifold we are interested in, we only need to ask for invariance under holomorphic transformations.

The best way to do the expansion is to use normal coordinates [41, 42] and consider fluctuations as

$$x^\alpha = x_0^\alpha + \xi^\alpha - \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma + \frac{1}{6}\left(2\Gamma^\alpha_{\beta\gamma}\Gamma^\beta_{\phi\delta} - \partial_\gamma\Gamma^\alpha_{\phi\delta}\right)\xi^\gamma\xi^\delta\xi^\phi + \dots \tag{4.5}$$

where $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\alpha'}\left(g_{\beta\alpha',\gamma} + g_{\gamma\alpha',\beta} - g_{\beta\gamma,\alpha'}\right)$ is the connection. Expanding a generic tensor we get (here we follow [46])

$$\frac{\partial x^{\alpha_1}}{\partial \xi^{\beta_1}} \cdots \frac{\partial x^{\alpha_l}}{\partial \xi^{\beta_l}} T_{\alpha_1\dots\alpha_l} = T_{\beta_1\dots\beta_l} + T_{\beta_1\dots\beta_l;\alpha}\xi^\alpha + \frac{1}{2}\left\{T_{\beta_1\dots\beta_l;\alpha\beta} + \frac{1}{3}\sum_i R^{\alpha_i}{}_{\alpha\beta\beta_i} T_{\beta_1\dots\alpha_i\dots\beta_l}\right\}\xi^\alpha\xi^\beta + \dots \tag{4.6}$$

where $R^\alpha{}_{\beta\gamma\delta}$ is the Riemann tensor and the semicolon ";" indicates covariant derivative. In particular

$$\frac{\partial x^\beta}{\partial \xi^\alpha} A_\beta = A_\alpha + A_{\alpha;\beta}\xi^\beta + \frac{1}{2}\left\{ A_{\alpha;\beta\gamma} + \frac{1}{3} R^\delta{}_{\beta\gamma\alpha} A_\delta \right\} \xi^\beta \xi^\gamma + \dots \tag{4.7}$$

Also we get from [46]

$$\frac{\partial \xi^\beta}{\partial x^\alpha} \partial_j x^\alpha = \partial_j x_0^\beta + D_j \xi^\beta + \frac{1}{3} R^\beta{}_{\gamma\delta\alpha}\, \partial_j x_0^\alpha\, \xi^\gamma \xi^\delta + \dots \tag{4.8}$$

where we defined the covariant derivative

$$D_j \xi^\beta = \partial_j \xi^\beta + \Gamma^\beta_{\alpha\gamma}\, \partial_j x_0^\alpha\, \xi^\gamma \tag{4.9}$$

Finally, we introduce a set of vielbeins $e_a^\alpha(x_0)$ such that $\delta^{ab} e_a^\alpha e_b^\beta = g^{\alpha\beta}$ and fluctuations ξ^a through

$$\xi^\beta = e_a^\beta\, \xi^a \tag{4.10}$$

which leads to the covariant derivative

$$D_j \xi^a = \partial_j \xi_a + \omega_\alpha{}^a{}_b\, \partial_j x^\alpha\, \xi^b \tag{4.11}$$

Here $\omega_\alpha{}^a{}_b$ is the usual spin connection defined as

$$\omega_\alpha{}^a{}_b = e_\beta^a\, e_{b;\alpha}^\beta \tag{4.12}$$

Since the vielbein is arbitrary, the classical action has an $SO(2n)$ gauge invariance. Again, since usually a vielbein is not globally defined, the perturbation procedure should also be $SO(2n)$-gauge invariant so that one can transition between different patches with vielbeins differing by arbitrary rotations. In our case however we need only to respect a $U(n)$ gauge invariance since the vielbeins are divided into holomorphic and antiholomorphic and we consider only rotations that do not mix them.

Putting all together we expand the action to second order as

$$S = S^{(0)} + S^{(2)} \tag{4.13}$$

with

$$\begin{aligned} S^{(2)} = {} & \frac{i\mu}{2} \int F_{ab} \xi^b D_t \xi^a + \frac{i\mu}{2} \int F_{\alpha a;b} \xi^a \xi^b \dot{x}_0^\alpha \\ & + \frac{\lambda}{2} \int D_j \xi^a D_j \xi^a + \frac{\lambda}{2} \int R_{\alpha ab\beta}\, \partial_j x_0^\alpha \partial_j x_0^\beta\, \xi^a \xi^b \end{aligned} \tag{4.14}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength associated with A_α and $F_{ab} = e_a^\alpha e_b^\beta F_{\alpha\beta}$. We note that the first order term in the action vanishes since there is no overlap between fast and slow modes.

To proceed we are going to consider the particular case we are interested in where

$$A_\mu = i\partial_\mu K,\ A_{\bar{\mu}} = -i\partial_{\bar{\mu}} K \quad \Rightarrow \quad F_{\mu\bar{\nu}} = -2i\partial_{\mu\bar{\nu}} K = -2ig_{\mu\bar{\nu}} \tag{4.15}$$

This means that

$$F_{a\bar{b}} = -2i\delta_{a\bar{b}} \tag{4.16}$$

where we split a into holomorphic and antiholomorphic indices $a, \bar{a}$ (see below eq.(2.2). Furthermore, from (4.15) we obtain:

$$F_{\alpha a;b} = 0 \tag{4.17}$$

as we knew since $F_{\alpha\beta}$ is actually the Kähler form. Using this, the quadratic action reduces to

$$S^{(2)} = 2\mu \int \xi^{\bar{a}} \partial_t \xi^a + 2\mu \int \omega_{\alpha\bar{a}b} \xi^{\bar{a}} \xi^b \dot{x}_0^\alpha + \lambda \int D_j \xi^{\bar{a}} D_j \xi^a \tag{4.18}$$

$$+\frac{\lambda}{2} \int \left(R_{\mu a\bar{b}\bar{\nu}} + R_{\bar{\nu}a\bar{b}\mu} + R_{\mu\bar{b}a\bar{\nu}} + R_{\bar{\nu}\bar{b}a\mu}\right) \partial_j x_0^\mu \partial_j x_0^{\bar{\nu}} \xi^a \xi^{\bar{b}} \tag{4.19}$$

To proceed we should introduce a cut-off in a gauge invariant way. For example we can use a proper time regularization [47] to define the inverse and determinant of the quadratic operator. From a more modern perspective we can probably use dimensional regularization in the spatial directions although it is not clear then how to treat the time direction. After doing the gauge invariant regularization, the spin connection ${\omega_\alpha}^a{}_b$, which is the $U(n)$ gauge field, can appear only in gauge invariant combinations. If we call its field strength (which is esentially the Riemann tensor) $\mathcal{F}$ then, due to rotational invariance, at first sight we can only have combinations such as $\mathcal{F}_{0j}\mathcal{F}_{0j}$ or $\mathcal{F}_{ij}\mathcal{F}_{ij}$ which are higher order in time or spatial derivatives. However, one can see that a possible gauge invariant term in the action is $\int {\omega_\alpha}^a{}_a\, \dot{x}^\alpha$ which is gauge invariant since ${\omega_\alpha}^a{}_a$ is a $U(1)$ gauge field, corresponding to the $U(1)$ factor in the holonomy group $U(n) = SU(n) \times U(1)$ (where n is the complex dimension of the manifold). In a general manifold we can attempt to consider a similar term $\int \Gamma^\alpha_{\beta\alpha}\, \dot{x}^\alpha$ but that fails since it is a total derivative $\int \partial_t \ln g = 0$. The term we discussed is invariant because we restrict ourselves to holomorphic transformations.

Therefore, at one loop, using a gauge invariant regularization, is equivalent to replacing $D_j \to \partial_j$ in the third term of $S^{(2)}$. From the D_t part however we get the second term whose expectation value is precisely the gauge invariant term that we have just discussed.

Now we do a Fourier transform (we assume the background fields are approximately constant from the point of view of the fast variables ξ^a):

$$\xi^a(x,t) = \int \frac{d^dk}{(2\pi)^d}\frac{d\omega}{2\pi}\,\xi^a(k,\omega)\,e^{ikx+i\omega t} \tag{4.20}$$

and get the ξ^a propagator

$$\langle \xi^{\bar a}(k_1,\omega_1)\xi^b(k_2,\omega_2)\rangle = \frac{(2\pi)^{d+1}}{\lambda k_1^2 + 2i\mu\omega_1}\delta^{\bar a b}\delta(k_1-k_2)\delta(\omega_1-\omega_2) \tag{4.21}$$

With this we compute $\langle S^{(2)}\rangle$. At this point we get delta functions evaluated at zero in momentum space that can be interpreted as spatial and time integrations of the background fields, namely $\delta_k(0)\delta_w(0) = \int \frac{d^dx\,dt}{(2\pi)^{d+1}}$. The result is

$$\langle S^{(2)}\rangle = 2\mu\mathcal{C}\int d^dx\,dt\,\omega_\alpha{}^a{}_a\,\dot x_0^\alpha - \lambda\mathcal{C}\int d^dx\,dt\,R_{\mu\bar\nu}\,\partial_j x_0^\mu\partial_j x_0^{\bar\nu} \tag{4.22}$$

where $\mathcal{C}$ is the integral over a thin momentum shell ($b\to 0^+$):

$$\mathcal{C} = \int_{\Lambda e^{-b}}^{\Lambda} \frac{d^dk d\omega}{(2\pi)^{(d+1)}}\frac{1}{\lambda k^2 + 2i\mu\omega} \simeq_{b\to 0} \frac{1}{(4\pi)^{\frac{d+1}{2}}}\frac{1}{2\mu\Lambda^{-d}}\frac{\Gamma(\frac{2+d}{4})}{\Gamma(\frac{d}{2})\Gamma(1+\frac{d}{4})}\,b = \bar{\mathcal{C}}b \tag{4.23}$$

is the momentum integral and we defined $\bar{\mathcal{C}} = \lim_{b\to 0}(\mathcal{C}/b)$. The cut-off was introduced through

$$(\lambda^2k^4 + \mu^2\omega^2) \leq \lambda^2\Lambda^4 \tag{4.24}$$

so that Λ is interpreted as a momentum cut-off ($k\leq\Lambda$). The energy cut-off is $\omega \leq \frac{\mu}{\lambda}\Lambda^2$. As expected we see the correction to be of order $1/(\mu\Lambda^{-d})$ as we discussed in the previous section.

Therefore, ignoring all terms higher order in derivatives (since we are considering a low energy expansion of the effective action), the one-loop terms effectively shift the metric and gauge field to

$$\tilde g_{\mu\bar\nu} = g_{\mu\bar\nu} - \mathcal{C}R_{\mu\bar\nu} \tag{4.25}$$

$$\tilde A_\mu = A_\mu + 2i\,\mathcal{C}\,\omega_\mu{}^a{}_a \tag{4.26}$$

$$\tilde A_{\bar\nu} = A_{\bar\nu} - 2i\,\mathcal{C}\,\omega_{\bar\mu}{}^{\bar a}{}_{\bar a} \tag{4.27}$$

Now we use that

$$R_{\mu\bar\nu} = -\partial_{\mu\bar\nu}\ln\det g_{\rho\bar\sigma} \tag{4.28}$$

and

$$\omega_\alpha{}^a{}_a = e^a_\nu\,e^\nu_{a;\alpha} = e^a_\nu\,\partial_\alpha e^\nu_a + \Gamma^\beta_{\alpha\beta} = -i\partial_\alpha\phi + \frac{1}{2}\partial_\alpha\ln\det g_{\rho\bar\sigma} \tag{4.29}$$

where ϕ in an undetermined phase that depends on the choice of vielbein through $\det e^a_\nu = e^{i\phi}\sqrt{\det g_{\rho\bar\sigma}}$. Fortunately, when replacing in the action, ϕ appears in a total derivative: $\int \partial_t \phi = 0$ (actually this is the statement that such term is $U(1)$ gauge invariant). Using now the fact that $A_\mu = i\partial_\mu K$ and $g_{\mu\bar\nu} = \partial_{\mu\bar\nu}K$ we see that the action has exactly the same form (2.2) but with a Kähler potential

$$\tilde K = K + \mathcal{C} \ln \det g_{\rho\bar\sigma} \tag{4.30}$$

Now we go to the second step and rescale the coordinates $x_j \to e^b x_j$ and $t \to e^{2b} t$ in order to restore the cut-off to its original value Λ (namely $k \to e^{-b}k$ so the new k extends to Λ instead of $e^{-b}\Lambda$). This is the standard cut and stretch procedure that zooms into the low energy region. From the action we see that this just rescales K by e^{bd}. So we get

$$\tilde K = K + b\left(\bar{\mathcal{C}} \ln \det g_{\rho\bar\sigma} + Kd\right) \tag{4.31}$$

where $\bar{\mathcal{C}} = \mathcal{C}/b$, *i.e.* we extract the b factor. We can then write a β-function as

$$\beta_K = \partial_b K = \bar{\mathcal{C}} \ln \det g_{\rho\bar\sigma} + Kd \tag{4.32}$$

In terms of the metric we have

$$\partial_b g_{\alpha\beta} = -\bar{\mathcal{C}} R_{\alpha\beta} + d\, g_{\alpha\beta} \tag{4.33}$$

For a relativistic sigma model this type of equation is well-known. The only point we needed here is that the term linear in time derivatives also renormalizes in the same way and the action has an invariant form[3]. This equation has an Einstein metric as a fixed point. If we are away from the fixed point, the metric flows. The first thing to note is that the volume changes under such flow since:

$$\partial_b \sqrt{g} = \frac{1}{2}\sqrt{g} g^{\alpha\beta}\partial_b g_{\alpha\beta} = \frac{1}{2}\sqrt{g}\left(-\bar{\mathcal{C}} R + d(2n)\right) \tag{4.34}$$

where R is the Ricci scalar. For the volume $V = \int \sqrt{g}$, we get

$$\frac{1}{V}\partial_b V = \frac{1}{2}\left(-\bar{\mathcal{C}} r + d(2n)\right), \quad \text{with} \quad r = \frac{1}{V}\int \sqrt{g} R \tag{4.35}$$

Then, as a final step, we can rescale K in such a way that the volume is fixed. Rescaling $K \to e^{\chi b} K$ with $\chi = \frac{r}{2n}\bar{\mathcal{C}} - d$ the flow equation becomes

$$\partial_b g_{\alpha\beta} = \bar{\mathcal{C}}\left(-R_{\alpha\beta} + \frac{r}{2n}\, g_{\alpha\beta}\right) \tag{4.36}$$

[3]We should note also that the relativistic calculation is special in two dimensions since then the classical action is scale invariant and the second step, namely rescaling the coordinates, does not change the metric. Therefore, the second term on the right hand side of (4.33) is absent in that case.

which is the so-called normalized Ricci flow. In the case of a compact Kähler manifold, under this flow, the metric is known to flow to a Kähler-Einstein metric [48]. However, we are interested also in the non-compact case of which we consider an example in the next section.

Finally, to compensate for the rescaling of K we have to rescale μ and λ getting

$$\beta_\mu = -\chi\mu \tag{4.37}$$

$$\beta_\lambda = -\chi\lambda \tag{4.38}$$

We see that the ratio μ/λ is fixed. We only change an overall scale in the action. Since $\bar{\mathcal{C}}$ is small where the one-loop approximation is valid we have $\chi < 0$. That means that the overall factor in the action grows and the system becomes classical. This is just the naive idea that as we lower the cut-off there are more spins in the minimal volume and therefore the system is more classical. The one-loop result is a small correction.

To summarize, the final result is that as we lower the cut-off the system becomes more classical and the metric flows to an Einstein metric.

Note that in the $SU(2)$ case (2.1) the metric of the sphere is already Einstein so nothing happens except that the system becomes classical in the infrared as follows from the simple reasoning already explained. However, in general the problem remains non-trivial. For example if we would want to compute the finite temperature partition function, then, at low temperature we need the partition function of a classical ferromagnet which is a non-trivial problem already studied for example by Polyakov using Wilson's renormalization group in [45]. However, in the case of interest for the reduced string action we have $d = 1$ and the classical partition function can actually be computed simply by considering the problem as the evaluation of a quantum mechanical propagator where the eigenstates are spherical harmonics. The result is

$$Z_{\text{cl.}}(\beta) = \int \mathcal{D}\vec{n}\, e^{-\frac{\beta\lambda}{2}\int_0^L (\partial_\sigma \vec{n})^2\, d\sigma} = \sum_{\ell=0}^{\sqrt{\beta\lambda\Lambda}} (2\ell+1) e^{-\frac{L}{\beta\lambda}\ell(\ell+1)} \tag{4.39}$$

From the field theory point of view, this partition function determines the distribution of the lowest anomalous dimensions of very long operators in the $SU(2)$ sector.

5. A particular case

When studying strings propagating in $AdS_5 \times Y^{p,q}$ a sigma model of the type we consider in this paper appears. The complex manifold has two complex dimensions $z_{1,2}$ with an $SU(2)$ isometry. The Kähler potential depends on $\rho = \bar{z}_1 z_1 + \bar{z}_2 z_2$. Parameterizing the

complex coordinates as

$$z_1 = \sin\frac{\theta}{2}\, e^{-\frac{i}{2}(\beta-\phi)}\sqrt{\rho} \tag{5.1}$$

$$z_2 = \cos\frac{\theta}{2}\, e^{-\frac{i}{2}(\beta+\phi)}\sqrt{\rho} \tag{5.2}$$

the action becomes

$$\begin{aligned} S = i\mu \int d\sigma dt K_x \left(\dot{\beta} + \cos\theta\dot{\phi}\right) \\ +\frac{1}{4}\lambda \int d\sigma dt \left\{K_x\left((\partial_\sigma\theta)^2 + \sin^2\theta(\partial_\sigma\phi)^2\right) + K_{xx}dx^2 + K_{xx}\left(\partial_\sigma\beta + \cos\theta\partial_\sigma\phi\right)^2\right\} \end{aligned} \tag{5.3}$$

where it turned out to be convenient to use $x = \ln\rho$ as a coordinate. From the positivity of the metric we see that we need $K_x > 0$ and $K_{xx} > 0$. We can now compute

$$\det g_{\rho\bar{\sigma}} = e^{-2x}K_x K_{xx} \tag{5.4}$$

and therefore the RG equation is

$$\partial_b K(x) = \bar{\mathcal{C}}\left(-2x + \ln K_x + \ln K_{xx} + \frac{d}{\bar{\mathcal{C}}}K\right) \tag{5.5}$$

where we used eq.(4.33) since now the manifold is non-compact and it is not clear that normalizing the volume is appropriate. In the case studied in [39], the manifold has orbifold singularities at $\rho = 0, \infty$. Therefore we are in the non-compact case and no rigorous mathematical result applies (at least that we know of). The type of singularity is such that K behaves as

$$K \simeq_{x\to\pm\infty} \eta_\pm x + A_\pm e^{\alpha_\pm x} + \mathcal{O}\left(e^{2\alpha_\pm x}\right) \tag{5.6}$$

Using this in the RG equation we can see that $\alpha_\pm$ are not renormalized. The slopes $\eta_\pm$ however, change according to:

$$\partial_b \eta_\pm = \bar{\mathcal{C}}\left(\alpha_\pm - 2 + \frac{d}{\bar{\mathcal{C}}}\eta_\pm\right) \tag{5.7}$$

The fixed point is at $\eta_\pm = -\frac{\bar{\mathcal{C}}}{d}(\alpha_\pm - 2)$. The problem is, however, that if the slopes $\eta_\pm$ are not at the fixed point then they flow away from it. This means that we have to start with the correct value of $\alpha_\pm$ and $\eta_\pm$. By fixing the metric at the end points, we expect that the system behaves as in the compact case and flows to the Kähler-Einstein metric although we did not verify that explicitely.

If we study the simplified model proposed in [39] it turns out that the constraints on the slopes are not satisfied[4]. This means that the model, or at least the way in which the continuum limit was taken, was perhaps too naive.

The general idea however seems correct in the sense that an infinite class of models flow to the IR fixed point implying that most of the details of the gauge theory disappear. This is specially true in the compact case. Therefore we conclude that the (classical) string action captures only the main features of the dilatation operator.

6. Conclusions

Motivated by some ideas described in [39], in this paper we studied the one-loop renormalization of certain actions given in terms of a Kähler potential K. We showed that, as expected, when we lower the cut-off, the system behaves more classically and K flows to a Kähler-Einstein metric. This follows from a known mathematical result for Kähler-Ricci flows on compact manifolds. In the case of the reduced action, non-compact cases also appear and we suggested that by putting appropriate boundary conditions, the same result should follow. However a more detailed analysis of this point is desirable.

One point to emphasize is that we consider the action from a low energy effective action point of view. The action has infinite number of terms and we consider only the lowest ones in an expansion in derivatives. In this sense we study the renormalization group of only those terms and ignore the others which are irrelevant (at least in the case $d = 1$ that we are interested in).

From a generic perspective, this action computes the anomalous dimensions of long operators in the large-N limit of a field theory. We see that, when the operators are very long, and we concentrate in the lowest anomalous dimensions, the effective action is determined by a fixed point and therefore is largely independent of the details of the theory. The hope is then that the large N-limit of a gauge theory could be independent of a detailed evaluation of the planar diagrams and should be given by statistical considerations (at least when dealing with large number of fields or particles).

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[4]It should be noted however that in those cases the volume is finite even if the manifold has singularities so we also expect the system to be similar to the compact case after we impose conditions near the singularities.

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